Analysis 1B — Tutorial 2

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# Introduction

Here is the material to accompany the 2nd Analysis 1B Tutorial on the 13th February. Alternative formats can be downloaded by clicking the download icon at the top of the page. Please send any comments or corrections to [Christian Jones (caj50)](mailto:caj50@bath.ac.uk). To return to the homepage, click [here](http://caj50.github.io/tutoring.html).

# Lecture Recap

The course has finally begun proper! For the next few weeks, we’re going to see the first section: limits and continuity of functions. Despite another complicated definition to work with, it will turn out that many of the concepts can be made eerily similar to stuff you saw from Semester 1.

## Definition of a Limit

### Functions

To begin, we pick up where Analysis 1A left off, and start talking about functions.

Definition 1.1 (Function)

Let and be sets. A function from to is a rule that assigns to each element of a unique element of . The set is called the domain of the function, and is called the codomain of the function.

So, to specify a function, we need a domain, codomain and a rule that tells us how to get between the two. We can write a function symbolically as follows:

A specific type of domain we are going to need in the near future is that of a *punctured neighbourhood*.

Definition 1.2 (Punctured Neighbourhood)

Let , and . Then is a punctured neighbourhood of if there exists a such that the set

For example, if we take , potential punctured neighbourhoods include:

* (This works with any )
* (This works with any )
* (This works with *any* positive )

### The Definition

One thing we may be interested in for a function is what happens as we approach a particular value from inside the domain. More specifically, we are interested in what happens to the corresponding codomain values: do they approach something finite, oscillate wildly, or behave even more bizarrely? In the first case, we can talk about a limiting value, of which the precise definition is formulated below:

Definition 1.3 (Limit of a Function)

Let , and let be a function defined on a punctured neighbourhood of . Then means

Alternatively, we can say that as if this definition is satisfied. Loosely, this says that “however close you want the function to be to at , you can always find a range of values that make this possible.” Graphically, we may be in a situation as in Figure 1.1.

Figure 1.1: A diagram showing the limiting process for functions. For the given \epsilon in the diagram, we see that whenever we are within a distance \delta of c in the domain, evaluating the function leaves us within a distance \epsilon of L. Note that the function isn’t defined at c, but the limiting value L still exists.

Figure 1.1: A diagram showing the limiting process for functions. For the given in the diagram, we see that whenever we are within a distance of in the domain, evaluating the function leaves us within a distance of . Note that the function isn’t defined at , but the limiting value still exists.

Once we have a definition, it makes sense to also consider its negation. Namely, we have that does not approach[[1]](#footnote-31) as when

### Using the Definition in Practice

In the exam, it is **highly** likely that you’ll be made to use this definition to calculate a limit. In doing so, you’ll need to bound a function that is wrapped up in an absolute value. There are two results from Semester 1 that can help with this, and if you haven’t already, make sure that you commit these results to memory.

Theorem 1.1 (Triangle Inequalities)

For :

* The second result here — the *reverse triangle inequality* — will be incredibly useful when we need to bound fractions.

## Two Useful Results

One thing we would definitely like to know is that if a limit of a function exists, is it unique? Luckily, the answer to this is yes, and is summed up in the following result:

Proposition 1.2 (Uniqueness of Limits)

Let and let be a function defined on a punctured neighbourhood of . If such that and , then .

Another useful result regarding functions we can state here is known as the *inertia principle*:

Proposition 1.3 (Inertia)

Let and let be a function defined on a punctured neighbourhood of . If , then for any , there exists such that

This idea is summed up in Figure 1.2 below. This principle is especially useful when the limit of a function at a point is positive, as it says that there is a punctured neighbourhood around in which the function remains positive.

Figure 1.2: A diagram showing the inertia principle in action. This principle says that if a function approaches a limit at a point, there is always an interval in which the function stays within a specified distance of the limit

Figure 1.2: A diagram showing the inertia principle in action. This principle says that if a function approaches a limit at a point, there is always an interval in which the function stays within a specified distance of the limit

## Sequential Characterisation of Limits

Using an - definition of the limit is not the only way we can define the limit of a function! In fact, you’ve already seen something similar when dealing with sequential continuity last semester.

Theorem 1.4 (Sequential Characterisation of Limit)

Let and let , with a punctured neighbourhood of . Then the following are equivalent[[2]](#footnote-43):

* For any sequence in which converges to ,

Why should you be pleased about this theorem? This allows you to apply all your Semester 1 knowledge of sequences to functions (for example, the algebra of limits)! In particular, if you can find two sequences and in converging to , but the sequences and approach different limits, this gives you an easy way of proving that function limits do not exist.

# Hints

In this section, you’ll find hints for the current week’s problem sheet. Try and have a go without them first, but hopefully these will help you solve the problems.

1. This one is mostly similar to the examples seen in lectures and the one we did in the tutorial. You might have to make a choice of delta near the start of your argument though, just so that is well defined.
2. Try making a clever choice of epsilon in the definition of limit. Then you should be able to rearrange and find the required , and .
3. Again, this is fairly similar to the ones we did in tutorials. Some Semester 1 material may also help here!

1. This can also be written . [↑](#footnote-ref-31)
2. You may see this abbreviated to ‘TFAE’ in some contexts. [↑](#footnote-ref-43)